Kinetic Equations

Solution to the Exercises

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Teachers: Prof. Chiara Saffirio, Dr. Théophile Dolmaire

Assistant: Dr. Daniele Dimonte – daniele.dimonte@unibas.ch

Exercise 1

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Assume that φ is **collision invariant**, i.e.

$$\varphi\left(v'\right) + \varphi\left(v'_*\right) = \varphi\left(v\right) + \varphi\left(v_*\right) \tag{1}$$

for all $v,\ v_* \in \mathbb{R}^3$, $\omega \in \mathbb{S}^2$, and with v' and v'_* defined as:

$$\begin{cases} v' = v + (v_* - v) \cdot \omega\ \omega, \\ v'_* = v_* - (v_* - v) \cdot \omega\ \omega. \end{cases} \tag{2}$$

- Assume additionally that φ vanishes on $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,0,0)$. Prove that φ is zero on $\mathbb{Z}^3$.
- Under the same assumption of the previous point, prove that actually φ is zero on $\mathbb{R}^3$.

  Hint: Denote $a = (1/2, 1/2, 0)$, $b = (1/2, -1/2, 0)$, $c = (-1/2, -1/2, 0)$ and $d = (-1/2, 1/2, 0)$, what can be said about $\varphi(a) + \varphi(b)$, $\varphi(b) + \varphi(c)$, $\varphi(c) + \varphi(d)$, $\varphi(d) + \varphi(a)$? What about $\varphi(a) + \varphi(c)$? Iterate this idea and use continuity to conclude.
- Consider now a generic continous φ which is collision invariant. Use the previous point to prove that there exist $a,\ c \in \mathbb{R}$ and $b \in \mathbb{R}^3$ such that

  $$\varphi(v) = a\left|v\right|^2 + b \cdot v + c, \tag{3}$$

  for any $v \in \mathbb{R}^3$.

Remark. *Notice that despite the similarities with the result presented in class, the final result is here achieved under much less regularity assumptions.*

Proof. Let $e_1 := (1,0,0)$, $e_2 := (0,1,0)$ and $e_3 := (0,0,3)$. Initially assume that $\varphi : \mathbb{R}^3 \to \mathbb{R}$, is collision invariant and vanishes on $0,\ e_1,\ e_2,\ e_3,\ -e_1$.

Consider the set $\Lambda := \left\{v \in \mathbb{R}^3 |\ \varphi(v) = 0\right\}$. By hypotheses $0,\ e_1,\ e_2,\ e_3,\ -e_1 \in \Lambda$. In the first part we prove that $\mathbb{Z}^3 \subseteq \Lambda$.

Step 1: Recall that we saw in class that the middle point between v and v_* and the middle point between v' and v'_* coincide. We will indicate that point as v_M in the following of the proof. Moreover we have that $|v - v_*| = |v' - v'_*|$. As we discussed in class, any value of ω is associated to one and only one pair $v',\ v'_*$ such that the middle point coincides with v_M and $|v - v_*| = |v' - v'_*|$. We will use this geometric characterization to complete our proof.

Step 2: We first prove that $e_1+e_2 \in \Lambda$. Indeed we have that if $v=e_1$, $v_*=e_2$, $v'=0$ and $v'_*=e_1+e_2$ we have $v+v_*=e_1+e_2=v'+v'_*$; on the other hand $|v-v_*|=|e_1-e_2|=\sqrt{2}=|e_1+e_2|=|v'-v'_*|$. This means that $\varphi(v'_*)=\varphi(v)+\varphi(v_*)-\varphi(v')=0$ and therefore $e_1+e_2 \in \Lambda$. More in general, if we prove that $v,\ v_*,\ v' \in \Lambda$, then $v'_* \in \Lambda$.

Step 3: Next, consider $v=-e_1,\ v_*=e_1,\ v'=e_2$ and $v'_*=-e_2$. We get $v+v_*=0$ and $v'+v'_*=0$, while at the same time $|v-v_*|=2=|v'-v'_*|$. Therefore $v'_*=-e_2 \in \Lambda$.

Analogously $v=e_1,\ v_*=-e_2,\ v'=0$ and $v'_*=e_1-e_2$. We get $v+v_*=e_1-e_2$ and $v'+v'_*=e_1-e_2$, while at the same time $|v-v_*|=\sqrt{2}=|v'-v'_*|$. Therefore $v'_*=e_1-e_2 \in \Lambda$.

Step 4: Given that the crucial ideas are the one introduced in the previous points, we just sketch the next steps. From the fact that $-e_1,\ e_1,\ e_1-e_2 \in \Lambda$ then $-e_1-e_2 \in \Lambda$; from the fact that $-e_1,\ e_1,\ e_1+e_2 \in \Lambda$ then $-e_1+e_2 \in \Lambda$.

Now, using iteratively that from the fact that $(n-1)\,e_1,\ ne_1+e_2,\ ne_1-e_2 \in \Lambda$ we get $(n+1)\,e_1 \in \Lambda$ and that from the fact that $(n-1)\,e_1,\ (n-1)\,e_1 \pm e_2,\ (n+1)\,e_1 \in \Lambda$ we get $(n+1)\,e_1 \pm e_2 \in \Lambda$ we get $\{ne_1+me_2|\ n \in \mathbb{N},\ m \in \{0,\ \pm 1\}\} \subseteq \Lambda$.

Proceeding similarly in all directions this implies that $\{ne_1+me_2|\ n,\ m \in \mathbb{Z}\} \subseteq \Lambda$.

Step 5: Consider now $v=e_1,\ v_*=-e_1,\ v'=e_3$ and $v'_*=-e_3$. We get $v+v_*=0=v'+v'_*$ and $|v-v_*|=2=|v'-v'_*|$. Therefore $v'_*=-e_3 \in \Lambda$.

Proceeding as before we first deduce that $e_3 \pm e_1,\ -e_3 \pm e_1,\ e_3 \pm e_2,\ -e_3 \pm e_2 \in \Lambda$. This implies that $\{nv_1+mv_2|\ n,\ m \in \mathbb{Z},\ v_1, v_2 \in \{e_1,\ e_2,\ e_3\}\}$.

Step 6: Finally from the previous step we deduce that $\mathbb{Z}^3 \subseteq \Lambda$.

For the next part, consider $a,\ b,\ c,\ d$ as defined in the hint. Consider $v=a=\frac{1}{2}e_1+\frac{1}{2}e_2$, $v_*=b=\frac{1}{2}e_1-\frac{1}{2}e_2$, $v'=0$, $v'_*=e_1$; we then get $v+v_*=e_1=v'+v'_*$ and that $|v-v_*|=1=|v'-v'_*|$. This implies that $\varphi(a)+\varphi(b)=\varphi(0)+\varphi(e_1)=0$. As a consequence $\varphi(a)=-\varphi(b)$. Proceeding analogously we get $\varphi(b)=-\varphi(c)=\varphi(d)$

Moreover, consider $v=a$, $v_*=c=-\frac{1}{2}e_1-\frac{1}{2}e_2$, $v'=b$ and $v'_*=-\frac{1}{2}e_1+\frac{1}{2}e_2$. We get $v+v_*=0=v'+v'_*$ and that $|v-v_*|=\sqrt{2}=|v'-v'_*|$. As a consequence we get $\varphi(a)+\varphi(c)=\varphi(b)+\varphi(d)$ and therefore $\varphi(a)=\varphi(b)=\varphi(c)=\varphi(d)=0$. This allows us to conclude that from the fact that $\{ne_1+me_2|\ n,\ m \in \mathbb{Z}\} \subseteq \Lambda$ we get $\left\{ne_1+me_2|\ n,\ m \in \frac{1}{2}\mathbb{Z}\right\} \subseteq \Lambda$. Iterating this we get that $\left\{ne_1+me_2|\ \exists k \in \mathbb{N},\ n,\ m \in \frac{1}{2^k}\mathbb{Z}\right\} \subseteq \Lambda$. Finally by continuity we get that $\{xe_1+ye_2|\ x,\ y \in \mathbb{R}\} \subseteq \Lambda$ proceeding similarly in all directions we can conclude that $\Lambda=\mathbb{R}^3$ and therefore $\varphi=0$.

For the final point, consider a generic continuous collision invariant function φ and define the function $\widetilde{\varphi}(v) := a\,|v|^2+b \cdot v+c$ such that $\widetilde{\varphi}=\varphi$ on the points $0,\ \pm e_1,\ e_2,\ e_3$. This correspond to five equations with five unknown; there exist then $a,\ c \in \mathbb{R},\ b \in \mathbb{R}^3$ such that $\varphi-\widetilde{\varphi}$ vanishes on $0,\ \pm e_1,\ e_2,\ e_3$, which is a continuous collision invariant. From the previous points now $\varphi-\widetilde{\varphi}=0$ and this implies the conclusion.

□

Exercise 2

Let (X, Σ, μ) be a finite measure space. Let $f : X \to X$ a **measure-preserving transformation**, i.e., a mapping such that for any $A \in \Sigma$ we have $\mu\left(f^{-1}(A)\right) = \mu(A)$. For any $x \in X$ and $A \in \Sigma$, we say that x is **recurrent with respect to** A if $\left|\left\{k \in \mathbb{N} \mid f^k(x) \in A\right\}\right| = +\infty$, where $f^{k+1}(x) := f\left(f^k(x)\right)$.

Prove the **Poincaré recurrence Theorem**, i.e., prove that for any measurable set $A \in \Sigma$ almost every point of A is recurrent with respect to A.

Discuss then how this would seem to contradict the H-theorem (this is the so-called Zermelo's Paradox).

Hint: Consider the family of sets $U_p := \bigcup_{k \geqslant p} f^{-k}(A)$. Can we express the set of non-recurrent points in term of $\{U_p\}_{p \in \mathbb{N}}$?

Proof. As in the hint define for any $p \in \mathbb{N}$ the set $U_p := \bigcup_{k \geqslant p} f^{-k}(A)$; clearly $f^{-1}(U_p) = U_{p+1}$ and therefore $\mu(U_p) = \mu(U_0) \geqslant \mu(A)$. Moreover we get $U_p = f^{-p}(U_0)$. Consider now the set of all the points in A which are not recurrent with respect to A; we get

$$
\begin{aligned}
\{x \in A \mid x \text{ is not recurrent w.r.t. } A\} &= \{x \in A \mid \ |\{n \in \mathbb{N} \mid f^n(x) \in A\}| < +\infty\} && (4)\\
&= A \backslash \{x \in A \mid \ |\{n \in \mathbb{N} \mid f^n(x) \in A\}| = +\infty\} && (5)\\
&= A \backslash \left(\bigcap_{p \in \mathbb{N}} (A \cap U_p)\right) = A \backslash \bigcap_{p \in \mathbb{N}} U_p && (6)\\
&= \bigcup_{p \in \mathbb{N}} (A \backslash U_p). && (7)
\end{aligned}
$$

From the definition we get $A \subseteq U_0$ and therefore $A \backslash U_p \subseteq U_0 \backslash U_p = U_0 \backslash f^{-p}(U_0)$. Given that $f^{-p}(U_0) \subseteq U_0$ we get

$$
0 \leqslant \mu(A \backslash U_p) \leqslant \mu\left(U_0 \backslash f^{-p}(U_0)\right) = \mu(U_0) - \mu\left(f^{-p}(U_0)\right) = \mu(U_0) - \mu(U_0) = 0. \quad (8)
$$

Therefore we get $\mu(A \backslash U_p)$ which implies $\mu(\{x \in A \mid x \text{ is not recurrent w.r.t. } A\}) = 0$.

□

Exercise 3

We will now study a toy model, useful to understand the Zermelo's Paradox.

Consider the following setting. We have N points on a ring, with N a large integer number. At every point there is a ball, that can be either white or black. Between every couple of points there is an edge that can contain or not contain a marker. We consider that the system evolves in discrete times according to the following rule: at each step, the balls rotate of one position (the ball in position 1 goes to position 2, 2 to 3 and so on, and finally the ball in position N goes to position 1). If the ball encounters a marker on the edge, it changes its color.

- Let $W(t)$ the total number of white balls and $B(t)$ the total number of black balls at time t. Let $w(t)$ (and $b(t)$ respectively) the number of white (and respectively black) balls that will cross a marker at the next step.

  Let $\Delta(t) := B(t) - W(t)$. Describe $\Delta(t+1)$ in terms of $\Delta(t)$.

  Let in addition μ be the fraction of markers over the total number of edges. Assume moreover that $\frac{w(t)}{W(t)} = \frac{b(t)}{B(t)} = \mu$ (this corresponds to the Stosszahlansatz). Find an explicit formula for $\Delta(t)$ in terms of $\Delta(0)$ and μ.

- Denote with $X_j(t)$ the color of the ball at position j at time t, where $X_j(t) = 1$ if the ball is black and $X_j(t) = -1$ if the ball is white. Denote with m_j the fact that a marker is or is not on the edge between position j and position $j+1$, where $m_j = 1$ if there is no marker (and the ball does not change colour) while $m_j = -1$ if there is a marker (and the ball does change colour).

  Describe $\Delta(t)$ in terms of $\{X_j(0)\}_{j=1}^N$ and $\{m_j\}_{j=1}^N$.

- Suppose now that every edge has a probability $0 \leqslant \mu \leqslant 1$ of having a marker. Denote with $\langle \cdot \rangle$ the expectation over all the possible configurations of markers, meaning that if M is the set of all the possible configurations and we denote with $m = \{m_j\}_{j=1}^N$ one such configuration, for $f : M \to \mathbb{R}$, we define

$$\langle f \rangle := \frac{1}{|M|} \sum_{m \in M} f(m). \tag{9}$$

  Given that for any $t > 0$ the value $\Delta(t)$ depends on the configuration of markers on the edges, we can calculate its expectation.

  Prove that for any $t < N$, $\langle \Delta(t) \rangle = (1-2\mu)^t \Delta(0)$.

- Discuss the link between the quantity we just obtained and the H-theorem for large values of N.

- Notice that the evolution of the system is reversible, that $\Delta(t)$ is periodic and find the period. Discuss how this solves the Zermelo's Paradox.

Proof. First of all, we get that the number of black balls at time $t+1$ is given adding the number of white balls that cross a marker to become black and substracting the number of black balls that cross a marker to become white to the number of black balls at time t, i.e., $B(t+1) = B(t) - b(t) + w(t)$. Similarly, for the white balls we get $W(t+1) = W(t) - w(t) + b(t)$. Therefore we get

$$\begin{aligned} \Delta(t+1) = B(t+1) - W(t+1) &= B(t) - b(t) + w(t) - (W(t) - w(t) + b(t)) &(10)\\ &= \Delta(t) - 2(b(t) - w(t)). &(11) \end{aligned}$$

If we now assume that $\frac{w(t)}{W(t)} = \frac{b(t)}{B(t)} = \mu$ we get

$$\Delta(t+1) = \Delta(t) - 2(b(t) - w(t)) = \Delta(t) - 2\mu(B(t) - W(t)) = (1-2\mu)\Delta(t). \tag{12}$$

Iterating the previous formula we get $\Delta(t) = (1-2\mu)^t \Delta(0)$.

Now, define $X_j(t)$ as in the text. We get that $\Delta(t) = \sum_{j=1}^{N} X_j(t)$. Also, by definition of $\{X_j(t)\}_{j=1}^{N}$ and $\{m_j\}_{j=1}^{N}$ we get $X_{j+1}(t+1) = m_j X_j(t)$ for all $j > 0$, while $X_1(t+1) = m_N X_N(t)$. Extend now the definition of m_j and X_j for any $j \in \mathbb{Z}$ periodically. In this way we get that iterating the formula for $X_j(t+1)$ we get

$$X_j(t) = \left(\prod_{k=j-t}^{j-1} m_k\right) X_{j-t}(0) = \left(\prod_{k=1}^{t} m_{j-k}\right) X_{j-t}(0). \tag{13}$$

As a consequence, this implies

$$\Delta(t) = \sum_{j=1}^{N} \left(\prod_{k=1}^{t} m_{j-k}\right) X_{j-t}(0) = \sum_{j=1}^{N} \left(\prod_{k=1}^{t} m_{j+t-k}\right) X_j(0). \tag{14}$$

First notice that $X_j(0)$ does not depend on $\{m_j\}_{j=1}^{N}$, therefore

$$\langle \Delta(t) \rangle = \sum_{j=1}^{N} \langle \left(\prod_{k=1}^{t} m_{j+t-k}\right) \rangle X_j(0). \tag{15}$$

We can now explicitly calculate the last term; indeed first of all given that we are looking at an average, the product depends only on the fact that we consider t different markers, not which one we consider, so we immediately get

$$\langle \prod_{k=1}^{t} m_{j+t-k} \rangle = \langle \prod_{k=1}^{t} m_k \rangle. \tag{16}$$

Moreover, the product is 1 if we have an even number of markers, -1 if we have an odd one. Therefore, if we indicate with $p_l(t)$ the probability of finding l markers on t consecutive edges, we get

$$\langle \prod_{k=1}^{t} m_k \rangle = \sum_{l=0}^{t} (-1)^l p_l(t). \tag{17}$$

Now we have that an edge has probability μ of having a marker, hence if $t < N$

$$p_l(t) = \binom{t}{l} \mu^l (1-\mu)^{t-l}. \tag{18}$$

We can then conclude

$$\langle \prod_{k=1}^{t} m_{j+t-k} \rangle = \sum_{l=0}^{t} (-1)^l p_l(t) = \sum_{l=0}^{t} (-1)^l \binom{t}{l} \mu^l (1-\mu)^{t-l} = (1-2\mu)^t. \tag{19}$$

We finally get

$$\langle \Delta(t) \rangle = \sum_{j=1}^{N} \langle \left(\prod_{k=1}^{t} m_{j+t-k}\right) \rangle X_j(0) = (1-2\mu)^t \Delta(0). \tag{20}$$

It is easy to see that $\Delta(t)$ is periodic, indeed $\Delta(2N) = \Delta(0)$ (every ball encounters every marker twice). Furthermore, $2N$ is the period if the number of marker is odd, while N is if the number of markers is even.

$\square$